Stochastic Processes and their Applications 8 (1979) 349–355

DERIVATIVES OF REGULARLY VARYING FUNCTIONS IN $\mathbf{R}^d$ AND DOMAINS OF ATTRACTION OF STABLE DISTRIBUTIONS*

L. de HAAN

Faculteit der Economische Wetenschappen, Erasmus Universiteit, Rotterdam, The Netherlands

S.I. RESNICK

Department of Statistics, Colorado State University, Fort Collins, CO 80523, U.S.A.

Received 1 March 1978
Revised 28 September 1978

In $\mathbf{R}^2$ the integral of a regularly varying (RV) function f is regularly varying only if f is monotone. Generalization to $\mathbf{R}^2$ of the one-dimensional result on regular variation of the derivative of an RV-function however is straightforward. Applications are given to limit theory for partial sums of i.i.d. positive random vectors in $\mathbf{R}^2_+$.

Multivariate regular variation	multivariate stable laws
Tauberian theorems	domains of attraction

0. Introduction

A function $f:\mathbf{R}_+\to\mathbf{R}_+$ whose derivative is regularly varying with exponent ρ is itself regularly varying with exponent $\rho+1$ The converse is true if the derivative is monotone and $\rho\neq-1$. This together with Karamata's Abelian and Tauberian theorems for Laplace transforms can be used to derive a limit theory for partial sums of i.i.d. positive random variables (see [1] sections XIII 5 and 6). We extend the results to higher-dimensional Euclidean space. The appropriate Abelian and Tauberian theorem for Laplace transforms is provided by Stam [7]. For simplicity we only formulate the theory in $\mathbf{R}^2$.

* The bulk of this work was carried out while S. Resnick was a visiting fellow at Econometric Institute, Erasmus University, Rotterdam. Grateful acknowledgment is made for financial support and hospitality. The revision was undertaken while supported by NSF Grant MCS78-00915.

1. Integrals and derivatives of RV-functions in $\mathbf{R}^2$.

Slightly different from [7] but convenient for our purposes we say that the function $f:\mathbf{R}_+\times\mathbf{R}_+\to\mathbf{R}_+$ is regularly varying at infinity (RV) if for all $x_1, x_2>0$

$$\lim_{t\to\infty}\frac{f(tx_1, tx_2)}{f(t,t)}=\lambda(x_1,x_2) \tag{1}$$

exists where $\lambda(x_1,x_2)$ is positive for all $x_1,x_2>0$. Then λ satisfies $\lambda(ax_1,ax_2)=a^\rho\lambda(x_1,x_2)$ for some real ρ and $\lambda(1,1)=1$. We call ρ the *index* (we then say f is RV_ρ) and λ the *limit function* of f. In $\mathbf{R}^1$ the integral of an RV-function is again RV but in $\mathbf{R}^2$ some extra condition such as monotonicity is needed. A function $f:\mathbf{R}_+\times\mathbf{R}_+\to\mathbf{R}_+$ will be called *non-increasing* (*non-decreasing*) if f is non-increasing (non-decreasing) in each variable separately.

Lemma 1. *Suppose $f:\mathbf{R}_+\times\mathbf{R}_+\to\mathbf{R}_+$ is monotone and regularly varying with limit function λ. If $\rho>-1$, then $\int_0^1\int_0^1\lambda(u_1,u_2)\,\mathrm{d}u_1\,\mathrm{d}u_2<\infty$.*

Proof. The statement is obvious for non-decreasing λ. If λ is non-increasing (and hence $\rho\leqslant 0$)

$$\iint\limits_{\substack{0<u_1\leqslant 1\\0<u_2\leqslant 1\\u_1\leqslant u_2}}\lambda(u_1,u_2)\,\mathrm{d}u_1\,\mathrm{d}u_2\leqslant\int_0^1\int_0^{u_2}\lambda(u_1,u_1)\,\mathrm{d}u_1\,\mathrm{d}u_2$$

$$=\int_0^1\int_0^{u_2}u_1^\rho\,\mathrm{d}u_1\,\mathrm{d}u_2<\infty$$

and similarly for the other half of the integration region.

Remark 1. The lemma does not necessarily hold if one only assumes f is regularly varying. Consider the example $\lambda(x_1,x_2)=x_1^{\rho+2}x_2^{-2}$. Likewise if one merely assumes f RV, the convergence of the integral $\int_1^\infty\int_1^\infty(u_1,u_2)\,\mathrm{d}u_1\,\mathrm{d}u_2$ can for no value of ρ be guaranteed as is shown by the example $\lambda(x_1,x_2)=(x_1\wedge x_2)^\rho$.

Theorem 1. *Suppose $f:\mathbf{R}_+\times\mathbf{R}_+\to\mathbf{R}_+$ is monotone and RV_ρ with $\rho>-1$ and limit function λ. If $\int_0^1 f(v,v)\,\mathrm{d}v<\infty$, then*

$$\lim_{t\to\infty}\frac{\int_0^{tx_1}\int_0^{tx_2}f(u_1,u_2)\,\mathrm{d}u_1\,\mathrm{d}u_2}{t^2f(t,t)}=\int_0^{x_1}\int_0^{x_2}\lambda(u_1,u_2)\,\mathrm{d}u_1\,\mathrm{d}u_2 \tag{2}$$

so that the function $\int_0^{x_1}\int_0^{x_2} f(u_1, u_2)\,du_1\,du_2$ is $\mathrm{RV}_{\rho+2}$ *with limit function*

$$\frac{\int_0^{x_1}\int_0^{x_2} \lambda(u_1, u_2)\,du_1\,du_2}{\int_0^1\int_0^1 \lambda(u_1, u_2)\,du_1\,du_2}.$$

Remark 2. If f is monotone, the condition $\int_0^1 f(v, v)\,dv < \infty$ implies but is not equivalent to $\int_0^1\int_0^1 f(x, y)\,dx, dy < \infty$. Example: $f(x, y) = x^{-1/2}y^{-1/2}$.

Proof. The quotient in the left-hand side of (2) equals $\int_0^{x_1}\int_0^{x_2} f(tu_1, tu_2)/f(t, t)\,du_1\,du_2$. For non-decreasing f the integrand is bounded by $(f(t, t))^{-1}f(tx_1, tx_2)$ and the result follows by dominated convergence.

In the case f is non-increasing, it is convenient to use a variant of Fatou's lemma known as Pratt's lemma [4]: If $0 \le h_n \le g_n$ are real valued functions on some measure space, and $h_n \to h$, $g_n \to g$ and $\int g_n \to \int g$, then $\int h_n \to \int h$ provided $\int h < \infty$. Now

$$\begin{aligned} &\frac{f(tu_1, tu_2)}{f(t, t)} \iota_{\{0<u_1\le x_1,\, 0<u_2\le x_2,\, u_1\le u_2\}}(u_1, u_2) \\ &\le \frac{f(tu_1, tu_1)}{f(t, t)} \iota_{\{0<u_1\le x_1,\, 0<u_2\le x_2,\, u_1\le u_2\}}(u_1, u_2) \\ &\le \frac{f(tu_1, tu_1)}{f(t, t)} \iota_{\{0<u_1\le x_1,\, 0<u_2\le x_2\}}(u_1, u_2) \end{aligned}$$

and each of these functions converge as $t \to \infty$. By one dimensional regular variation (Karamata's Theorem—see [2] p. 15) the integral of the latter function goes to a limit:

$$\lim_{t\to\infty} \iint\limits_{\substack{0<u_1\le x_1\\ 0<u_2\le x_2}} \frac{f(tu_1, tu_1)}{f(t, t)}\,du_1\,du_2 = \lim_{t\to\infty} x_2 \int_0^{x_1} \frac{f(tu_1, tu_1)}{f(t, t)}\,du_1 = \frac{x_2 x_1^{\rho+1}}{\rho+1}.$$

Applying Pratt's lemma twice gives with the help of Lemma 1

$$\lim_{t\to\infty} \iint\limits_{\substack{0<u_1\le x_1\\ 0<u_2\le x_2\\ u_1\le u_2}} \frac{f(tu_1, tu_2)}{f(t, t)}\,du_1\,du_2 = \iint\limits_{\substack{0<u_1\le x_1\\ 0<u_2\le x_2\\ u_1\le u_2}} \lambda(u_1, u_2)\,du_1\,du_2$$

and similarly for the other half of the integration region.

Before proving the converse i.e. regular variation of the derivative of an RV-function we first state a lemma.

Lemma 2. *If* $f : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ *is monotone and* RV_ρ, *then the limit function* λ *is continuous on* $(0, \infty)^2$.

Proof. Suppose for concreteness that f is non-decreasing. For any vector $\boldsymbol{x}$ in the rectangle

$$\{\boldsymbol{x} \mid (1-\delta)\boldsymbol{x}_0 \leq \boldsymbol{x} \leq (1+\delta)\boldsymbol{x}_0\}$$

(the inequalities are componentwise) we have

$$(1-\delta)^{\alpha}\lambda(\boldsymbol{x}_0) = \lambda((1-\delta)\boldsymbol{x}_0) \leq \lambda(\boldsymbol{x}) \leq \lambda((1+\delta)\boldsymbol{x}_0) = (1+\delta)^{\alpha}\lambda(\boldsymbol{x}_0).$$

Theorem 2. *Suppose $G:\mathbf{R}_+ \times \mathbf{R}_+ \to \mathbf{R}_+$ is absolutely continuous with a positive and monotone density f. Suppose G is RV_ρ, $\rho > 1$, with a limit function λ which is absolutely continuous with density d. Then d can be taken continuous and*

$$\lim_{t\to\infty} \frac{t^2 f(tx_1, tx_2)}{G(t,t)} = d(x_1, x_2) \quad \text{for } x_1, x_2 > 0 \tag{3}$$

so that f is $\mathrm{RV}_{\rho-2}$ with limit function $(d(1,1))^{-1}d(x_1,x_2)$.

Proof. Suppose f is non-increasing. Consider for $x_1, x_2 > 0$, $0 < \varepsilon < x_1 \wedge x_2$

$$\int_{x_1-\varepsilon}^{x_1}\int_{x_2-\varepsilon}^{x_2} \frac{t^2 f(tu_1, tu_2)}{G(t,t)}\,du_1\,du_2 =$$

$$= \frac{G(t(x_1-\varepsilon), t(x_2-\varepsilon)) - G(t(x_1-\varepsilon), tx_2) - G(tx_1, t(x_2-\varepsilon)) + G(tx_1, tx_2)}{G(t,t)}.$$

By assumption the right-hand side tends to

$$\lambda(x_1-\varepsilon, x_2-\varepsilon) - \lambda(x_1-\varepsilon, x_2) - \lambda(x_1, x_2-\varepsilon) + \lambda(x_1, x_2).$$

The left-hand side is at least $t^2 f(tx_1, tx_2)\varepsilon^2/G(t,t)$. So

$$\overline{\lim_{t\to\infty}} \frac{t^2 f(tx_1, tx_2)}{G(t,t)} \leq \frac{\lambda(x_1-\varepsilon, x_2-\varepsilon) - \lambda(x_1-\varepsilon, x_2) - \lambda(x_1, x_2-\varepsilon) + \lambda(x_1, x_2)}{\varepsilon^2}.$$

Letting $\varepsilon \downarrow 0$ we get for Lebesgue-almost all $x_1, x_2 > 0$

$$\overline{\lim_{t\to\infty}} \frac{t^2 f(tx_1, tx_2)}{G(t,t)} \leq d(x_1, x_2).$$

Taking the rectangle $(x_1, x_1+\varepsilon] \times (x_2, x_2+\varepsilon]$ as the integration region we get similarly for almost all $x_1, x_2 > 0$

$$\underline{\lim_{t\to\infty}} \frac{t^2 f(tx_1, tx_2)}{G(t,t)} \geq d(x_1, x_2).$$

Hence (3) holds almost everywhere. Clearly if (3) holds for (x_1, x_2), then (3) also holds for (ax_1, ax_2) with $a > 0$ arbitrary and then $d(ax_1, ax_2) = a^{\rho-2}d(x_1, x_2)$ by the regular variation of G. Pick (x_0, y_0) with polar coordinates (r_0, θ_0). There exist $\theta_n \downarrow \theta_0$ such that (3) holds for (r, θ_n) with $r > 0$, $n \geq 1$. Define

$$x_n = (\tan\theta_n)^{-1} y_0, \qquad y_n = (\tan\theta_n) x_0$$

so that $x_n < x_0$, $y_n > y_0$. It follows that

$$\frac{d(x_0, y_n)}{d(x_n, y_0)} = \left(\frac{\tan \theta_n}{\tan \theta_0}\right)^{\rho-2} \to 1 \quad (n \to \infty).$$

Since d is monotone this gives us $d(x_0, y_0+) = d(x_0-, y_0)$.

By the monotonicity of f

$$\frac{t^2 f(tx_n, ty_0)}{G(t,t)} \geq \frac{t^2 f(tx_0, ty_0)}{G(t,t)} \geq \frac{t^2 f(tx_0, ty_n)}{G(t,t)}.$$

Letting $t \to \infty$ and then $n \to \infty$ gives

$$d(x_0, y_0) \geq \overline{\lim_{n\to\infty}} \frac{t^2 f(tx_0, ty_0)}{G(t,t)} \geq \underline{\lim_{n\to\infty}} \geq d(x_0, y_0+)$$

and since x_0, y_0 are arbitrarily chosen in $(0, \infty)^2$ we have that (3) holds everywhere. The continuity of d follows from Lemma 2. For nondecreasing f a similar proof applies.

2. Domains of attraction of stable probability distributions

Consider independent random vectors with positive components

$$(X_1^{(1)}, X_1^{(2)}), (X_2^{(1)}, X_2^{(2)}), \ldots$$

from the same probability distribution function F. Suppose there exist positive constants $(a_n)_{n=1}^{\infty}$ such that the sequence

$$\left(a_n^{-1} \sum_{i=1}^{n} X_i^{(1)}, a_n^{-1} \sum_{i=1}^{n} X_i^{(2)}\right) \tag{4}$$

converges to a limit distribution not concentrated at one point. For the Laplace transform $\hat{F}$ of F we then have

$$\lim_{n\to\infty} \hat{F}^n(a_n^{-1} t_1, a_n^{-1} t_2) = \hat{H}(t_1, t_2)$$

where $\hat{H}$ is the Laplace transform of H, the limit df of (4). Taking logarithms and applying $1 - \hat{F} \sim -\log \hat{F}$ as $\hat{F} \uparrow 1$ we get

$$\lim_{n\to\infty} n\{1 - \hat{F}(a_n^{-1} t_1, a_n^{-1} t_2)\} = -\log \hat{H}(t_1, t_2)$$

or by a familiar argument (cf. [2, p. 8])

$$\lim_{t\downarrow 0} \frac{1 - \hat{F}(tt_1, tt_2)}{1 - \hat{F}(t,t)} = \frac{-\log \hat{H}(t_1, t_2)}{-\log \hat{H}(1,1)} \tag{5}$$

The marginals of H are one-sided stable and at least one marginal is non-degenerate so the exponent of regular variation of $1-\hat{F}(1/t_1, 1/t_2)$ is in the open interval $(-1, 0)$.

We introduce the function

$$G(x_1, x_2) = \int_0^{x_1} \int_0^{x_2} (1-F(s_1, s_2))\, ds_1\, ds_2$$

and note that the Laplace transform $\hat{G}$ of G satisfies

$$\hat{G}(t_1, t_2) = \int_0^{\infty} \int_0^{\infty} e^{-t_1 x_1} e^{-t_2 x_2} G(dx_1, dx_2) = \frac{1-\hat{F}(t_1, t_2)}{t_1 t_2}$$

so by (5)

$$\lim_{t \downarrow 0} \frac{\hat{G}(tt_1, tt_2)}{\hat{G}(t, t)} = \frac{\log \hat{H}(t_1, t_2)/\log \hat{H}(1, 1)}{t_1 t_2}.$$

Hence $\hat{G}$ satisfies the conditions of Stam's [7] theorem 6 and by Stam's theorems 5 and 6 weak convergence of (4) is equivalent to regular variation of G. This by our theorems is equivalent to regular variation of $1-F$ with exponent $-\alpha$ $(0<\alpha<1)$.

By de Haan and Resnick [3] regular variation of $1-F$ is equivalent to

$$\lim_{t \to \infty} \frac{1-F(tx_1, tx_2)}{1-F(t, t)} = \nu(A^c(x_1, x_2))$$

where $A(x_1, x_2) = \{(y_1, y_2) \mid y_1 \leq x_1, y_2 \leq x_2\}$ and ν is a measure on $[0, \infty) \times [0, \infty)$ with $\nu(A^c(x_1, x_2)) < \infty$ if and only if $x_1 > 0$ and $x_2 > 0$. Moreover for $0 \leq \theta_1 \leq \theta_2 \leq \frac{1}{2}\pi$ and $R > 0$

$$\nu\{(x_1, x_2) \mid \theta_1 < \arctan x_2/x_1 \leq \theta_2, x_1^2 + x_2^2 \geq R^2\} = R^{-\alpha} S((\theta_1, \theta_2])$$

where S is a finite non-vanishing measure on $[0, \frac{1}{2}\pi]$. Easy calculations lead to

$$\hat{H}(t_1, t_2) = \exp \int_0^{\infty} \int_0^{\infty} (e^{-t_1 x_1 - t_2 x_2} - 1)\nu(dx_1, dx_2).$$

Domains of attraction of stable df's in the general case are treated in [6]; see also [5].

References

[1] W. Feller, An Introduction to Probability Theory and its Applications (Wiley, New York, 1966).

[2] L. de Haan, On Regular Variation and its Applications to the Weak Convergence of Sample Extremes, Tract 32, Mathematisch Centrum, Amsterdam (1970).

[3] L. de Haan and S.I. Resnick, Limit theory for multivariate sample extremes, Z. Wahrscheinlichkeitstheorie und Verw. Gebiete 40 (1977) 317–337.

[4] J. Pratt, On interchanging limits and integrals, Ann. Math. Statist. 31 (1960) 74–77.

[5] S.I. Resnick and P. Greenwood, A bivariate stable characterization and domains of attraction, Preprint, Statistics Department, Colorado State University (1978).

[6] E.L. Rvaçeva, On the domains of attraction of multidimensional distributions, Select. Translat. Math. Statist. Probab. 2 (1962) 183–207.
[7] A.J. Stam, Regular variation in $\mathbf{R}_+^d$ and the Abel–Tauber theorem, Preprint, Mathematisch Instituut, Rijksuniversiteit Groningen (1977).